Magic graphs and the faces of the Birkhoff polytope

MAYA MOHSIN AHMED
Department of Mathematics
University of California, Davis, CA 95616
UNITED STATES OF AMERICA
e-mail: maya@math.ucdavis.edu http://www.math.ucdavis.edu/˜maya

ABSTRACT
Magic labelings of graphs are studied in great detail by Stanley in [18] and [19], and Stewart in [20] and [21]. In this article, we construct and enumerate magic labelings of graphs using Hilbert bases of polyhedral cones and Ehrhart quasi-polynomials of polytopes. We define polytopes of magic labelings of graphs and digraphs. We give a description of the faces of the Birkhoff polytope as polytopes of magic labelings of digraphs.

KEY WORDS
Magic graphs; Polyhedral cones; Birkhoff polytope.

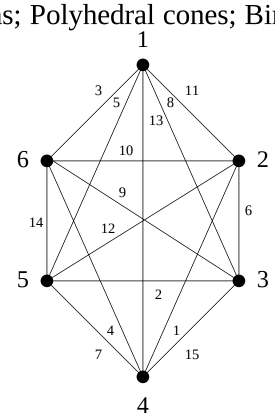

Figure 1: A magic labeling of K_6 [21].

1 Introduction.

Let G be a finite graph. A *labeling* of G is an assignment of a nonnegative integer to each edge of G. A *magic labeling of magic sum* r of G is a labeling such that for each vertex v of G the sum of the labels of all edges incident to v is the magic sum r (loops are counted as incident only once) [18, 19, 20, 21]. Graphs with a magic labeling are also called *magic graphs* (see Figure 1 for an example of a magic labeling of the complete graph K_6 of magic sum 40).

We define a *magic labeling* of a digraph D of *magic sum* r to be an assignment of a nonnegative integer to each edge of D, such that for each vertex v_i of D, the sum of the labels of all edges with v_i as the initial vertex is r, and the sum of the labels of all edges with v_i as the terminal vertex is also r. Thus magic labelings of a digraph is a network flow, where the flow into and out of every vertex, is the magic sum of the labeling. If we consider the labels of the edges of G as variables, the defining magic sum conditions are simply linear equations, and the set of magic labelings of G becomes the set of integral points inside a *pointed polyhedral cone* C_G [15]. Henceforth, we call C_G the cone of magic labelings of G. A *Hilbert basis* of C_G has the property that any magic labeling of G can be expressed as a linear combination with nonnegative integer coefficients of the elements of the Hilbert basis (see [1], [2], or [15]). An *irreducible magic labeling* of a graph is a magic labeling that cannot be written as a sum of two other magic labelings. The *minimal* Hilbert basis of C_G is the set of all irreducible magic labelings and is unique [15]. Henceforth, when we refer to the Hilbert basis in this article, we mean the minimal Hilbert basis.

Let $v_1, v_2, \ldots, v_n$ denote the vertices of G

and let $e_{i_1}, e_{i_2}, \ldots, e_{i_{m_i}}$ denote the edges of G that are incident to the vertex v_i of G. Consider the polytope

$$\mathcal{P}_G = \{L \in C_G \subseteq \mathbb{R}^q,\ \sum_{j=1}^{m_i} L(e_{i_j}) = 1; i = 1, \ldots, n\}.$$

We will refer to $\mathcal{P}_G$ as the polytope of magic labelings of G and denote $H_G(r)$ to be the number of magic labelings of G of magic sum r. Then, $H_G(r)$ is the *Ehrhart quasi-polynomial* of $\mathcal{P}_G$ [16].

We define a polytope $\mathcal{P}_D$ of magic labelings of a digraph D as follows. Let $e_{i_1}, \ldots, e_{i_{m_i}}$ denote the edges of D that have the vertex v_i as the initial vertex and let $f_{i_1}, f_{i_2}, \ldots, f_{i_{s_i}}$ denote the edges of D for which the vertex v_i is the terminal vertex, then:

$$\mathcal{P}_D = \{L \in C_D \subseteq \mathbb{R}^q,\ \sum_{j=1}^{m_i} L(e_{i_j}) = \sum_{j=1}^{s_i} L(f_{i_j}) = 1; i = 1, \ldots, n\}.$$

Let $H_D(r)$ denote the number of magic labelings of D of magic sum r. Then like before, $H_D(r)$ is the Ehrhart quasi-polynomial of $\mathcal{P}_D$. We now connect the magic labelings of digraphs to magic labelings of bipartite graphs.

Lemma 1.1. *For every digraph D, there is a bipartite graph G_D such that the magic labelings of D are in one-to-one correspondence with the magic labelings of G_D. Moreover, the magic sums of the corresponding magic labelings of D and G_D are also the same.*

Proof. Denote a directed edge of a digraph D with v_i as the initial vertex, and v_j as the terminal vertex, by e_{ij}. Let L be a magic labeling of D of magic sum r. Consider a bipartite graph G_D in $2n$ vertices, where the vertices are partitioned into two sets $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$, such that there is an edge between a_i and b_j, if and only if, there is an edge e_{ij} in D. Consider a labeling L_{G_D} of G_D such that the edge between the vertices a_i and b_j is labeled with $L(e_{ij})$. Observe that the sum of the labels of the edges incident to a_i is the same as the sum of the labels of incoming edges at the vertex v_i of

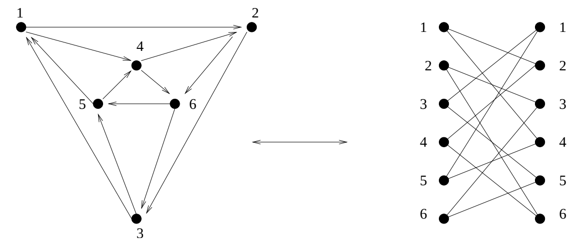

Figure 2: A digraph D_O and its corresponding bipartite graph G_{D_O}.

D. Also, the sum of the labels of edges at a vertex b_j is the sum of the labels of outgoing edges at the vertex v_j of D. Since L is a magic labeling, it follows that L_{G_D} is a magic labeling of G_D with magic sum r. Going back-wards, consider a magic labeling L' of G_D. We label every edge e_{ij} of D with the label of the edge between a_i and b_j of G_D to get a magic labeling L_D of D. Observe that L' and L_D have the same magic sum. Hence, there is a one-to-one correspondence between the magic labelings of D and the magic labelings of G_D. $\square$

For example, the magic labelings of the Octahedral digraph with the given orientation D_O in Figure 2 are in one-to-one correspondence with the magic labelings of the bipartite graph G_{D_O}.

A graph G is called a *positive graph* if for any edge e of G there is a magic labeling L of G for which $L(e) > 0$ [18]. Since edges of G that are always labeled zero for any magic labeling of G may be ignored to study magic labelings, we will concentrate on positive graphs in general. We use the following results by Stanley from [18] and [19] to prove Theorems 1.4 and 1.5 and Corollary 1.3.1.

Theorem 1.1 (Theorem 1.1, [19]). *Let G be a finite positive graph. Then either $H_G(r)$ is the Kronecker delta δ_{0r} or else there exist polynomials $I_G(r)$ and $J_G(r)$ such that $H_G(r) = I_G(r) + (-1)^r J_G(r)$ for all $r \in \mathbb{N}$.*

Theorem 1.2 (Theorem 1.2, [19]). *Let G be a finite positive graph with at least one edge. The degree of $H_G(r)$ is $q-n+b$, where q is the number of edges of G, n is the number of vertices, and b is*

the number of connected components of G which are bipartite.

Theorem 1.3 (Theorem 1.2, [18]). *Let G be a finite positive bipartite graph with at least one edge, then $H_G(r)$ is a polynomial.*

We now conclude that $H_D(r)$ is a polynomial for every digraph D.

Corollary 1.3.1. *If D is a digraph, then $H_D(r)$ is a polynomial of degree $q - 2n + b$, where q is the number of edges of D, n is the number of vertices, and b is the number of connected components of the bipartite graph G_D.*

Proof. The one-to one correspondence between the magic labelings of D and the magic labelings of G_D, implies by Theorem 1.3 that $H_D(r)$ is a polynomial, and by Theorem 1.2 that the degree of $H_D(r)$ is $q - 2n + b$, where b is the number of connected components of G_D that are bipartite. □

Consider the polytope $\mathcal{P} := \{x|Ax \leq b\}$. Let c be a nonzero vector, and let $\delta = \max\{cx|Ax \leq b\}$. The affine hyperplane $\{x|cx = \delta\}$ is called a *supporting hyperplane* of $\mathcal{P}$. A subset F of $\mathcal{P}$ is called a *face* of $\mathcal{P}$ if $F = \mathcal{P}$ or if F is the intersection of $\mathcal{P}$ with a supporting hyperplane of $\mathcal{P}$. Alternatively, F is a face of $\mathcal{P}$ if and only if F is nonempty and

$$F = \{x \in \mathcal{P}|A'x = b'\}$$

for some subsystem $A'x \leq b'$ of $Ax \leq b$. See [15] for basic definitions with regards to polytopes.

Therefore, a face of $\mathcal{P}_G$ is a polytope of the form

$$\{L \in \mathcal{P}_G, L(e_{i_k}) = 0; e_{i_k} \in E_0\},$$

where $E_0 = \{e_{i_1}, \ldots, e_{i_r}\}$ is a subset of the set of edges of G.

Theorem 1.4. *Let G be a finite positive graph with at least one edge. Then the polytope of magic labelings of G, $\mathcal{P}_G$ is a rational polytope with dimension $q-n+b$, where q is the number of edges of G, n is the number of vertices, and b is the number of connected components of G that are bipartite. The d-dimensional faces of $\mathcal{P}_G$ are the d-dimensional polytopes of magic labelings of positive subgraphs of G with n vertices and at most $n - b + d$ edges.*

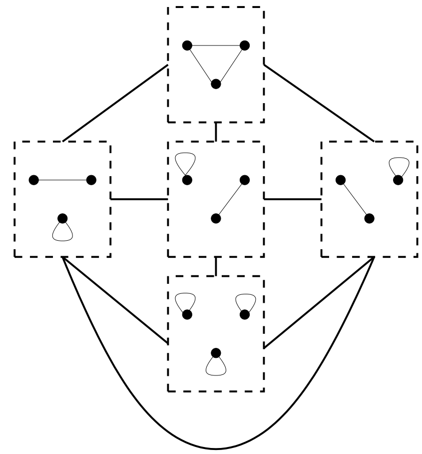

Figure 3: The edge graph of $\mathcal{P}_{\Gamma_3}$.

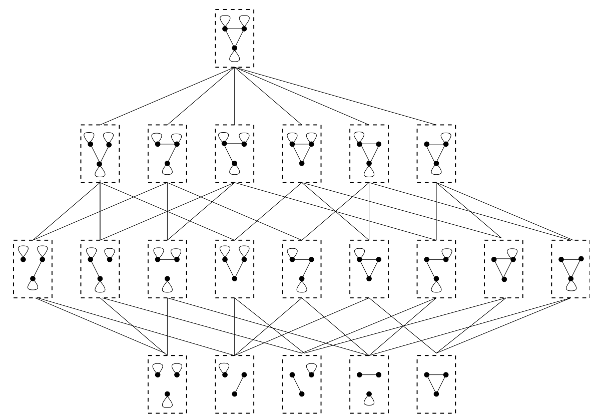

Figure 4: The face poset of $\mathcal{P}_{\Gamma_3}$.

Observe from Theorem 1.4 that there is an edge between two vertices v_i and v_j of $\mathcal{P}_G$ if and only if there is a graph with at most $n - b + 1$ edges, with magic labelings v_i and v_j. The edge graph of $\mathcal{P}_{\Gamma_3}$ is given in Figure 3. Similarly, we can draw the face poset of $\mathcal{P}_G$ (see Figure 4 for the face poset of $\mathcal{P}_{\Gamma_3}$).

An $n \times n$ *semi-magic square* of magic sum r is an $n \times n$ matrix with nonnegative integer en-

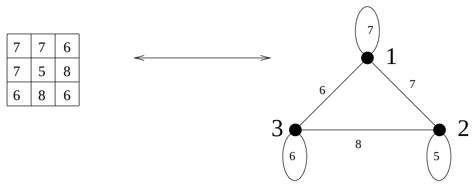

Figure 5: A magic labeling of Γ_3 and its corresponding symmetric magic square.

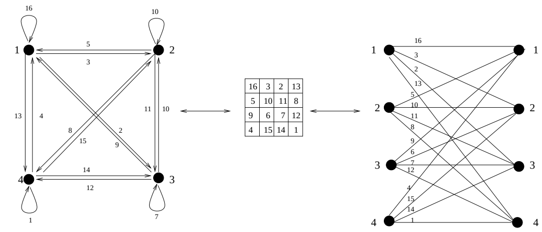

Figure 6: Two different graph labelings associated to a semi-magic square.

tries such that the entries of every row and column add to r. *Doubly stochastic matrices* are $n \times n$ matrices in $\mathbb{R}^{n^2}$ such that their rows and columns add to 1. The set of all $n \times n$ doubly stochastic matrices form a polytope B_n, called the *Birkhoff polytope*. See [7], [8], or [15] for a detailed study of the Birkhoff polytope.

A *symmetric magic square* is a semi-magic square that is also a symmetric matrix. Let $H_n(r)$ denote the number of symmetric magic squares of magic sum r (see [3], [19], and the references therein for the enumeration of symmetric magic squares). We define the polytope $\mathcal{S}_n$ of $n \times n$ symmetric magic squares to be the convex hull of all real nonnegative $n \times n$ symmetric matrices such that the entries of each row (and therefore column) add to one.

A one-to-one correspondence between symmetric magic squares $M = [m_{ij}]$ of magic sum r, and magic labelings of the graph Γ_n of the same magic sum r was established in [19]: let e_{ij} denote an edge between the vertex v_i and the vertex v_j of Γ_n. Label the edge e_{ij} of Γ_n with m_{ij}, then this labeling is a magic labeling of Γ_n with magic sum r. See Figure 5 for an example. Therefore, we get $\mathcal{P}_{\Gamma_n} = \mathcal{S}_n$ and $H_{\Gamma_n}(r) = H_n(r)$.

Corollary 1.4.1. *The polytope of magic labelings of the complete general graph $\mathcal{P}_{\Gamma_n}$ is an $n(n-1)/2$ dimensional rational polytope with the following description*

$$\mathcal{P}_{\Gamma_n} = \left\{ \begin{array}{l} L = (L(e_{ij}) \in \mathbb{R}^{\frac{n(n+1)}{2}}; \\ L(e_{ij}) \geq 0; \\ 1 \leq i,j \leq n, i \leq j, \\ \sum_{j=1}^{i} L(e_{ji}) + \sum_{j=i+1}^{n} L(e_{ij}) = 1 \\ \textit{for } i = 1, \ldots, n \end{array} \right\}$$

The d-dimensional faces of $\mathcal{P}_{\Gamma_n}$ are d-dimensional polytopes of magic labelings of positive graphs with n vertices and at most $n+d$ edges. There are $\binom{2n-1}{n}$ faces of $\mathcal{P}_{\Gamma_{2n}}$ that are copies of the Birkhoff polytope B_n.

We define a digraph D to be a *positive digraph* if the corresponding bipartite graph G_D is positive.

Theorem 1.5. *Let D be a positive digraph with at least one edge. Then, $\mathcal{P}_D$ is an integral polytope with dimension $q - 2n + b$, where q is the number of edges of D, n is the number of vertices, and b is the number of connected components of G_D that are bipartite. The d-dimensional faces of $\mathcal{P}_D$ are the d-dimensional polytopes of magic labelings of positive subdigraphs of D with n vertices and at most $2n - b + d$ edges.*

Let Π_n denote the complete digraph with n vertices, i.e, there is an edge from each vertex to every other, including the vertex itself (thereby creating a loop at every vertex), then G_{Π_n} is the the complete bipartite graph $K_{n,n}$. We get a one-to-one correspondence between semi-magic squares $M = [m_{ij}]$ of magic sum r and magic labelings of Π_n of the same magic sum r by labeling the edges e_{ij} of Π_n with m_{ij}. This also implies that there is a one-to-one correspondence between semi-magic squares and magic labelings of $K_{n,n}$ (this correspondence is also mentioned in [18] and [20]). See Figure 6 for an example.

A good description of the faces of Birkhoff polytope is not known [14]. We can now give an explicit description of the faces of the Birkhoff polytope.

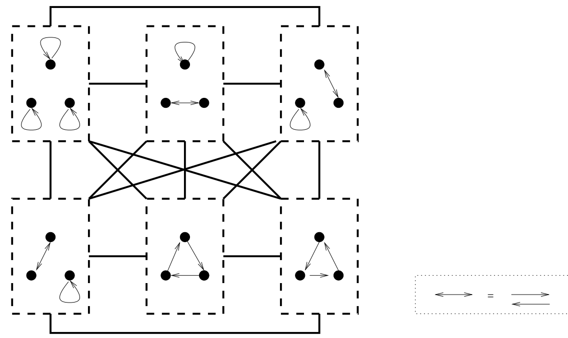

Figure 7: The edge graph of the Birkhoff Polytope B_3.

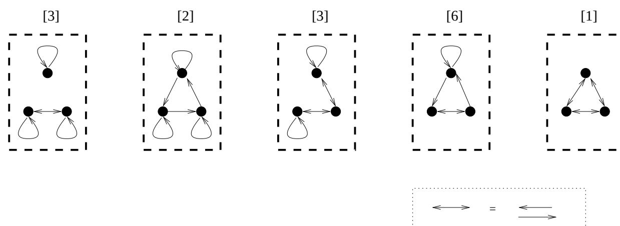

Figure 8: The generators of the edges of the Birkhoff Polytope B_3.

Corollary 1.5.1. *$\mathcal{P}_{\Pi_n}$ is the Birkhoff polytope B_n. The d-dimensional faces of B_n are polytopes of magic labelings of positive digraphs with dimension d, n vertices and at most $2n + d - 1$ edges. The vertices of $\mathcal{P}_D$, where D is a positive digraph, are permutation matrices.*

See Figure 7 for the edge graph of B_3. Two faces of a polytope of magic labelings of a graph (or a digraph) are said to be *isomorphic faces* if the subgraphs (subdigraphs, respectively) defining the faces are isomorphic. A set of faces is said to be a *generating set of d-dimensional faces* if every d-dimensional face is isomorphic to one of the faces in the set. See Figures 8, 9, 10, and 11 for the generators of the edges, the two dimensional faces, the facets, and the Birkhoff polytope B_3, respectively. The numbers in square brackets in the figures indicate the number of faces in the isomorphism class of the given face.

The proofs of Theorems 1.4 and 1.5, and Corollaries 1.4.1 and 1.5.1 are presented in Sec-

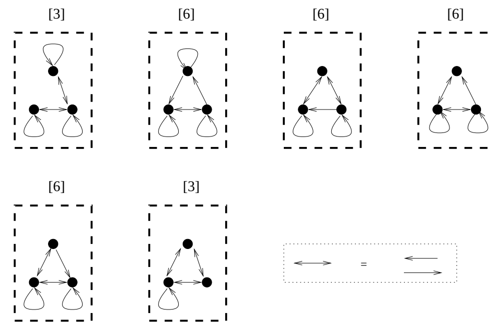

Figure 9: The generators of the 2-dimensional faces of the Birkhoff Polytope B_3.

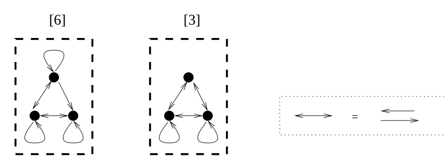

Figure 10: The generators of the facets of the Birkhoff Polytope B_3.

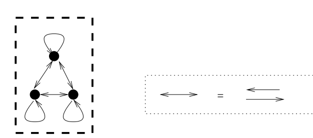

Figure 11: The Birkhoff Polytope B_3.

tion 2.

2 Polytopes of magic labelings.

A polytope $\mathcal{P}$ is called *rational* if each vertex of $\mathcal{P}$ has rational coordinates.

An element β in the semigroup S_{C_G} is said to be *completely fundamental*, if for any positive integer n and $\alpha, \alpha' \in S_{C_G}$, $n\beta = \alpha + \alpha'$ implies $\alpha = i\beta$ and $\alpha' = (n-i)\beta$, for some positive integer i, such that $0 \leq i \leq n$ (see [17]).

Lemma 2.1. *$\mathcal{P}_G$ is a rational polytope.*

Proof. Proposition $4.6.10$ of Chapter 4 in [17] states that the set of extreme rays of a cone and the set of completely fundamental solutions are identical. Proposition 2.7 in [18] states that every completely fundamental magic labeling of a graph G has magic sum 1 or 2. Thus, the extreme rays of the cone of magic labelings of a graph G are irreducible 2-matchings of G. We get a vertex of $\mathcal{P}_G$ by dividing the entries of a extreme ray by its magic sum. Thus, $\mathcal{P}_G$ is a rational polytope. □

Lemma 2.2. *The dimension of $\mathcal{P}_G$ is $q - n + b$, where q is the number of edges of G, n is the number of vertices, and b is the number of connected components that are bipartite.*

Proof. Ehrhart's theorem states that the degree of $H_G(r)$ is the dimension of $\mathcal{P}_G$ [6]. The degree of $H_G(r)$ is $q - n + b$ by Theorem 1.2. Therefore, the dimension of $\mathcal{P}_G$ is $q - n + b$. □

Lemma 2.3. *The d-dimensional faces of $\mathcal{P}_G$ are the d-dimensional polytopes of magic labelings of positive subgraphs of G with n vertices and at most $n - b + d$ edges.*

Proof. An edge e labeled with a zero in a magic labeling L of G does not contribute to the magic sum, therefore, we can consider L as a magic labeling of a subgraph of G with the edge e deleted. Since a face of $\mathcal{P}_G$ is the set of magic labelings of G where some edges are always labeled zero, it follows that the face is also the set of all the magic labelings of a subgraph of G with these edges deleted. Similarly, every magic labeling of a subgraph H with n vertices corresponds to a magic labeling of G, where the missing edges of G in H are labeled with 0. Now, let H be a subgraph such that the edges $e_{r1}, \ldots, e_{rm}$ are labeled zero for every magic labeling of H. Then the face defined by H is same as the face defined by the positive graph we get from H after deleting the edges $e_{r1}, \ldots, e_{rm}$. Therefore, the faces of $\mathcal{P}_G$ are polytopes of magic labelings of positive subgraphs.

By Lemma 2.2, the dimension of $\mathcal{P}_G$ is $q - n + b$. Therefore, to get a d-dimensional polytope, we need to label at least $q - n + b - d$ of G edges always 0. This implies that the d-dimensional face is the set of magic labelings of a positive subgraph of G with n vertices and at most $n - b + d$ edges. □

The proof of Theorem 1.4 follows from Lemmas 2.1, 2.2, and 2.3. We can now prove Corollary 1.4.1.

Proof of Corollary 1.4.1.

It is clear from the one-to-one correspondence between magic labelings of Γ_n and symmetric magic squares that $\mathcal{P}_{\Gamma_n}$ has the given description. Since the graph Γ_n has $\frac{n(n+1)}{2}$ edges and n vertices, and every graph is a subgraph of Γ_n, it follows from Theorem 1.4 that the dimension of $\mathcal{P}_{\Gamma_n}$ is $\frac{n(n-1)}{2}$; the d-dimensional faces of $\mathcal{P}_{\Gamma_n}$ are d-dimensional polytopes of magic labelings of positive graphs with n vertices and at most $n + d$ edges.

We can partition the vertices of Γ_{2n} into two equal sets A and B in $\binom{2n-1}{n}$ ways: Fix the vertex v_1 to be in the set A , then we can choose the n vertices for the set B in $\binom{2n-1}{n}$ ways, and the remaining $n - 1$ vertices will belong to the set A. By adding the required edges, we get a complete bipartite graph for every such partition of the vertices of Γ_{2n}. Thus, the number of subgraphs of Γ_{2n} that are isomorphic to $K_{n,n}$ is $\binom{2n-1}{n}$. Therefore, there are $\binom{2n-1}{n}$ faces of $\mathcal{P}_{\Gamma_{2n}}$ that are Birkhoff polytopes because every isomorphic copy of $K_{n,n}$ contributes to a face of $\mathcal{P}_{\Gamma_{2n}}$. □

Lemma 2.4. *Let G be a graph with n vertices. A labeling L of G with magic sum s can be lifted to*

a magic labeling L' of the complete general graph Γ_n with magic sum s.

Proof. Since G is a subgraph of Γ_n, every labeling L of G can be lifted to a labeling L' of Γ_n, where

$$L'(e_{ij}) = \begin{cases} L(e_{ij}) & \text{if } e_{ij} \text{ is also an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

Since the edges with nonzero labels are the same for both L and L', it follows that the magic sums are also the same. □

Lemma 2.5. *Let G be a graph with n vertices. The minimal Hilbert basis of C_G can be lifted to a subset of the minimal Hilbert basis of C_{Γ_n}.*

Proof. If L is an irreducible magic labeling of G, then clearly it lifts to an irreducible magic labeling L' of Γ_n. Since the minimal Hilbert basis is the set of all irreducible magic labelings, we get that the minimal Hilbert basis of C_G corresponds to a subset of the minimal Hilbert basis of C_{Γ_n}. □

Similarly, we can prove:

Lemma 2.6. *For a digraph D with n vertices, a magic labeling L with magic sum s can be lifted to a magic labeling L' of Π_n with the same magic sum s. The minimal Hilbert basis of C_D can be lifted to a subset of the minimal Hilbert basis of C_{Π_n}.*

Lemma 2.7. *Let D be a digraph with n vertices. All the elements of the minimal Hilbert basis of C_D have magic sum 1.*

Proof. It is well-known that the minimal Hilbert basis of semi-magic squares are the permutation matrices (see [15]) and therefore have magic sum 1. The one-to-one correspondence between magic labelings of Π_n and semi-magic squares implies that the minimal Hilbert basis elements of C_{Π_n} have magic sum 1. It follows by Lemma 2.6 that all the elements of the minimal Hilbert basis of C_D have magic sum 1. □

We now prove our results about polytope of magic digraphs.

Proof of Theorem 1.5. By Lemma 2.7, all the elements of the Hilbert basis of C_D have magic sum 1. Since the extreme rays are a subset of the Hilbert basis elements, it follows that the vertices of $\mathcal{P}_D$ are integral. Since $\mathcal{P}_D = \mathcal{P}_{G_D}$, it follows by Theorem 1.4 that the dimension of $\mathcal{P}_D$ is $q - 2n + b$; the d-dimensional faces of $\mathcal{P}_D$ are the d-dimensional polytopes of magic labelings of positive subdigraphs of D with n vertices and at most $2n - b + d$ edges. □

We derive our results about the faces of the Birkhoff polytope as a consequence.

Proof of Corollary 1.5.1. The one-to-one correspondence between semi-magic squares and magic labelings of Π_n gives us that $\mathcal{P}_{\Pi_n} = B_n$. Since every digraph with n vertices is a subgraph of Π_n, by Theorem 1.5, it follows that its d-dimensional faces are d-dimensional polytopes of magic labelings of positive digraphs with n vertices and at most $2n - 1 + d$ edges. Since the vertex set of a face of B_n is a subset of the vertex set of B_n it follows that the vertices of $\mathcal{P}_D$, where D is a positive digraph, are permutation matrices. □

3 Applications.

In this section, we present some examples and applications of magic graphs. Interesting examples of magic digraphs are Cayley digraphs of finite groups. Let G be a finite group $\{g_1, g_2, \dots, g_n = I\}$. The *Cayley group digraph* of G is a graphical representation of G: every element g_i of the group G corresponds to a vertex v_i $(i = 1, 2, \dots, n)$ and every pair of distinct vertices v_i, v_j is joined by an edge labeled with α where $g_\alpha = g_j g_i^{-1}$ [12]. For example, the Cayley digraph for the permutation group

$$S_3 = \{g_1 = (123), g_2 = (132), g_3 = (23), \\ g_4 = (12), g_5 = (13), g_6 = I\}$$

is given in Figure 12.

Proposition 3.1. *The Cayley digraph of a group of order n is a magic digraph with magic sum $\frac{n(n-1)}{2}$.*

Proof. Let e_{ij} denote an edge between the vertex v_i and v_j of the Cayley digraph such that v_i is the

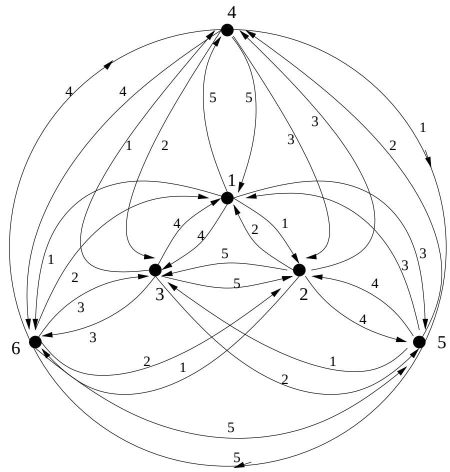

Figure 12: Cayley digraph of the group S_3 [12].

initial vertex and v_j is the terminal vertex. Let v_l be a vertex of the Cayley digraph, and let α be an integer in the set $\{1, 2, \ldots, n-1\}$. Let $g_p = g_\alpha g_l$ and let $g_q = g_l g_\alpha$. Then, the edges e_{lp} and e_{ql} are labeled by α. Also, $g_j g_i^{-1} = g_n = I$ if and only if $i = j$. Hence, a Cayley group digraph is a magic digraph with magic sum $1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}$ (see also Chapter 8, Section 5 in [12]). □

A digraph is called *Eulerian* if for each vertex v the indegree and the outdegree of v is the same. Therefore, Eulerian digraphs can also be studied as magic digraphs where all the edges are labeled by 1 (see [4] for the applications of Eulerian digraphs to digraph colorings). An n*-matching* of G is a magic labeling of G with magic sum at most n and the labels are from the set $\{0, 1, \ldots, n\}$ (see [13], chapter 6). A *perfect matching* of G is a 1-matching of G with magic sum 1.

Proposition 3.2. *The perfect matchings of G are the minimal Hilbert basis elements of C_G of magic sum 1 and the number of perfect matchings of G is $H_G(1)$.*

Proof. Magic labelings of magic sum 1 always belong to the minimal Hilbert basis because they are irreducible. Therefore, perfect matchings belong to the minimal Hilbert basis because they have magic sum 1. Conversely, every magic labeling of magic sum 1 is a perfect matching. So we conclude that the perfect matchings of G are the minimal Hilbert basis elements of C_G of magic sum 1. The fact that the number of perfect matchings of G is $H_G(1)$ follows by the definition of $H_G(1)$. □

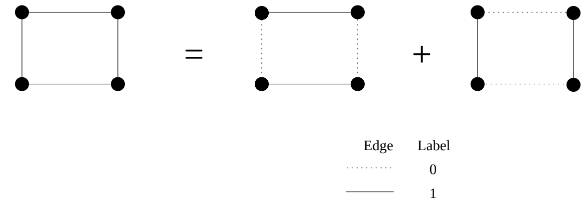

Figure 13: Graph Factorization.

Hilbert basis can also be used to study factorizations of labeled graphs. We define *Factors* of a graph G with a labeling L to be labelings $L_i, i = 1, \ldots, r$ of G such that $L(G) = \sum_{i=1}^{r} L_i(G)$, and if $L_i(e_k) \neq 0$ for some edge e_k of G, then $L_j(e_k) = 0$ for all $j \neq i$. A decomposition of L into factors is called a *factorization* of G. An example of a graph factorization is given in Figure 13. See Chapters 11 and 12 of [12] for a detailed study of graph factorizations.

Our results enable us to reprove some known facts about the Birkhoff polytope as well. For example, Theorem 1.5 gives us that the dimension of B_n is $(n-1)^2$. The leading coefficient of the Ehrhart polynomial of B_n is the volume of B_n. This number has been computed for $n = 1, 2, \ldots, 9$ (see [5] and [9]).

The software 4ti2 [11] can be used to find the Hilbert bases of C_G, and the software LattE [10] can be used to compute the generating functions of $H_G(r)$ effectively. See [3] for other results about magic labelings of graphs. See [22] for a study of other types of magic graphs.

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